(1) $x \neq 0$ のとき、$f(x)=\dfrac{\sqrt{2}x}{\sqrt{1+2x^2}}=\dfrac{\sqrt{2}}{\sqrt{\frac{1}{x^2}+2}}$ (単調増加) で $x\to\infty$ とすると $f(x)\to 1$

さらに $f(0)=0$ より

$y=\dfrac{\sqrt{2}x}{\sqrt{1+2x^2}}$ において $0\leqq y<1$.

これを x について解き、x と y を交換して、

$$y=\frac{x}{\sqrt{2(1-x^2)}}$$

$$\therefore\ f^{-1}(x)=\frac{x}{\sqrt{2(1-x^2)}}\quad (0\leqq x<1)$$

(2)

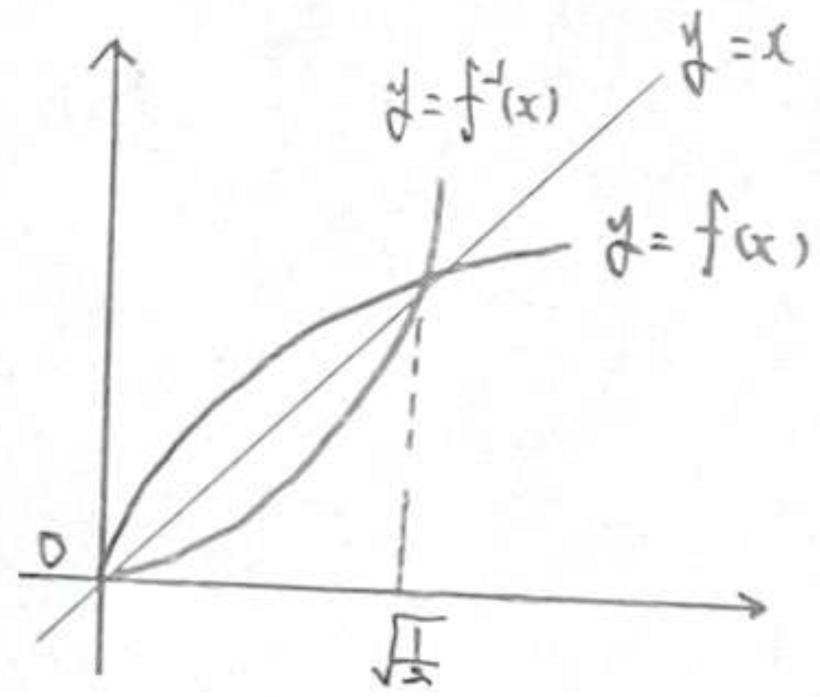
$$I=\int_0^{\alpha}\frac{dx}{x^2+\alpha^2}=\int_0^{\frac{\pi}{4}}\frac{1}{\alpha^2(1+\tan^2\theta)}\times\frac{\alpha}{\cos^2\theta}\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\alpha}=\frac{\pi}{4\alpha}$$

cf) $\dfrac{dx}{d\theta}=\dfrac{\alpha}{\cos^2\theta}$

x	$0\to\alpha$
θ	$0\to\frac{\pi}{4}$

(3) 逆関数の性質から、グラフの概形は右図の通り。



cf) $\dfrac{\sqrt{2}x}{\sqrt{1+2x^2}}=x$

$x(\sqrt{2}-\sqrt{1+2x^2})=0$

$\therefore\ x=0,\ \sqrt{\frac{1}{2}}$

また $f\left(\sqrt{\frac{1}{3}}\right)=\frac{\sqrt{10}}{5}>\sqrt{\frac{1}{3}}$

求める体積を V とすると、

$$\frac{V}{\pi}=\int_0^{\sqrt{\frac{1}{2}}}\{f(x)\}^2-\{f^{-1}(x)\}^2\,dx=\int_0^{\sqrt{\frac{1}{2}}}\left\{\frac{2x^2}{1+2x^2}-\frac{x^2}{2(1-x^2)}\right\}dx$$

ここで

$$\int_0^{\sqrt{\frac{1}{2}}}\frac{2x^2}{1+2x^2}\,dx=\int_0^{\sqrt{\frac{1}{2}}}\left(1-\frac{1}{1+2x^2}\right)dx=\sqrt{\frac{1}{2}}-\frac{\sqrt{2}}{4}\pi=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4}\pi$$

$$-\int_0^{\sqrt{\frac{1}{2}}}\frac{x^2}{2(1-x^2)}\,dx=\int_0^{\sqrt{\frac{1}{2}}}\left\{\frac{1}{2}+\frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\right\}dx=\frac{1}{2\sqrt{2}}+\frac{1}{4}\left(\log\left|\sqrt{\tfrac{1}{2}}-1\right|-\log\left|\sqrt{\tfrac{1}{2}}+1\right|\right)$$

$$=\frac{\sqrt{2}}{4}+\frac{1}{4}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\qquad \frac{\sqrt{2}}{4}+\frac{1}{2}\log(\sqrt{2}-1)\qquad \left(\because\ \frac{\sqrt{2}-1}{\sqrt{2}+1}=(\sqrt{2}-1)^2\right)$$

したがって

$$\frac{V}{\pi}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4}\pi+\frac{\sqrt{2}}{4}+\frac{1}{2}\log(\sqrt{2}-1)=\frac{3\sqrt{2}}{4}-\frac{\sqrt{2}}{4}\pi+\frac{1}{2}\log(\sqrt{2}-1)$$

$$\therefore\ V=\pi\left\{\frac{3\sqrt{2}}{4}-\frac{\sqrt{2}}{4}\pi+\frac{1}{2}\log(\sqrt{2}-1)\right\}$$